The Open University

MT365/Specimen

Course Examination
Graphs, networks and design

Time allowed: 3 hours

There are TWO parts to this paper.
55% of the available marks are assigned to Part 1 (5 marks per question) and 45% are assigned to Part 2 (15 marks per question).
In Part 1 you should attempt as many questions as you can. In Part 2 you should attempt THREE questions: ONE question from each section. Do NOT attempt more than one essay question. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.
Additional answer books are available from the invigilator, if needed.
Important notes:

- *Crossed-out work will NOT be marked, so you are advised not to cross anything out until you are sure that you have replaced it with something better.*
- *The QUESTION PAPER is NOT sent to the script markers, so you should not write anything on the question paper that you wish to be marked.*

In the wording of the questions, '*give*', '*state*', '*specify*' and '*write down*' do not require justification. Similarly, '*construct*', '*tabulate*', '*sketch*' and '*draw*' require only the final product. On the other hand, words such as '*apply*', '*determine*', '*explain*', '*find*', '*justify*', '*show*', '*use*' and '*verify*' mean that we require you to show your working or reasoning.

At the end of the examination

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.
Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

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Part 1

Part 1 carries 55% of the total marks for the examination (5 marks per question). Answer as many questions as you can from this part.

Please begin each Part 1 question on a new page.

Question 1

(a) Draw the digraph whose incidence matrix is

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

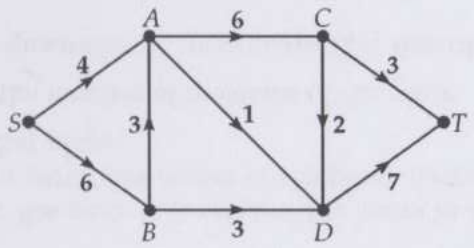
[2]

(b) Write down the out-degree and in-degree sequences for this digraph. [1]

(c) Check that the handshaking dilemma holds for this digraph. [2]

Question 2

Consider the following basic network in which the number next to each arc is the capacity of the arc:

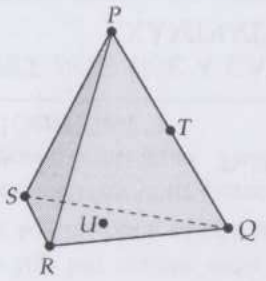


(a) Draw a diagram showing a maximum flow from S to T, clearly indicating the flow along each arc. [3]

(b) Write down, or indicate on your diagram, a minimum cut and state its capacity. [2]

Question 3

Consider the following figure, in which the points P, Q, R and S are the vertices of a tetrahedron, and the point U lies on the face QRS:



Describe the convex hull of each of the following sets:

(a) $\{Q, R, U\}$; [1]

(b) $\{U, T\}$; [1]

(c) $\{P, Q, R, S, T\}$; [1]

(d) $\{Q, R, S, T\}$; [1]

(e) $\{Q, R, S, U\}$. [1]

Question 4

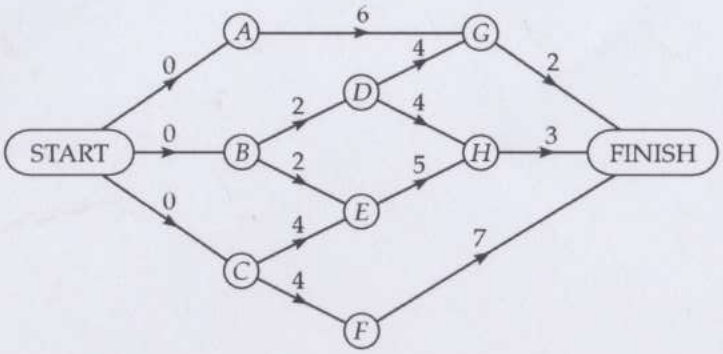
(a) There are six non-isomorphic unlabelled trees with 6 vertices. Draw them. [3]

(b) At least one of these unlabelled trees is central: indicate one that is. (No explanation required.) [1]

(c) How many *labelled* trees are there with 6 vertices? (Give a reason for your answer.) [1]

Question 5

Consider the following activity network. in which the vertices represent activities and the number next to each arc is the completion time (in weeks) of the activity represented by the vertex at the beginning of the arc:



(a) Write down:
   (i) the minimum completion time of the project, if an unlimited number of workers are available;
   (ii) the corresponding critical path. [2]

(b) Find the floats of activities D and B. [3]

Question 6

A thin triangular flat plate is constrained to move in its own plane in such a way that a particular corner of the plate always lies on a given straight line in that plane.

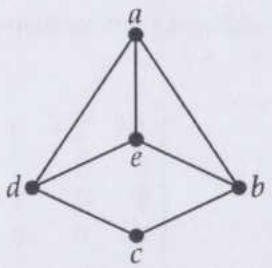
(a) Write down the number of freedoms of the plate. [1]

(b) List suitable quantities or coordinates that describe the pose of the plate. [2]

(c) Write down the number of further constraints needed to immobilize the plate, and describe one way of imposing these constraints. [2]

Question 7

Consider the following graph G:



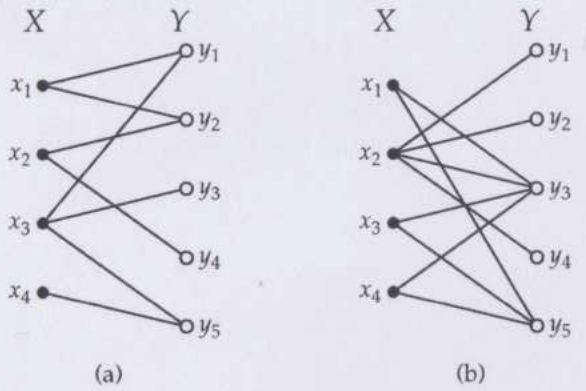
(a) Find the chromatic number $\chi(G)$; [1]

(b) Find the chromatic index $\chi'(G)$. [3]

(c) Write down the independence number ind (G). [1]

Question 8

For each of the following bipartite graphs, determine whether a maximum matching with 4 edges exists. If so, find such a matching; if not, write down a subset of X for which the marriage condition is not satisfied.



[5]

Question 9

Consider the code $C = \{0000,\ 0110,\ 1001,\ 1111\}$.

Decide whether the code C is:

(a) a linear code; [1]

(b) a cyclic code; [1]

(c) a self-dual code. [3]

(Give a brief reason in each case.)

Question 10

Two algorithms have time complexity functions

$$T_1(n) = n \log_2 n + 2n^{3/2} \quad \text{and} \quad T_2(n) = n^2.$$

Determine which of these algorithms is the faster:

(a) when $n = 4$; [3]

(b) when n is large. [2]

Question 11

(a) Show that $\{0, 2, 3, 7\}$ is a perfect difference set (modulo 13). [2]

(b) Write down the *first four* blocks of the cyclic design obtained from this difference set. [1]

(c) Verify the equation $\lambda(v - 1) = r(k - 1)$ for this cyclic design. [2]

Part 2

Part 2 is divided into three sections and carries 45% of the total marks for the examination. **You should attempt THREE questions from this part: one from each section. Do NOT attempt more than one essay question.**

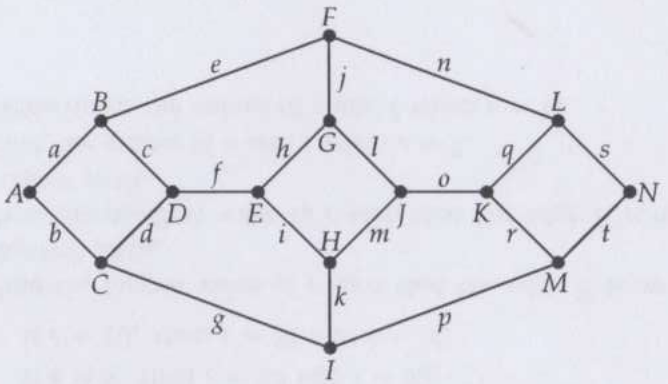
Each question in this part is assigned **15 marks**.

Please begin each Part 2 question on a new page.

SECTION A: GRAPHS

Question 12

The graph shown below has twelve vertices of degree 3 and two vertices of degree 2. The vertices are labelled with capital letters, and the edges are labelled with lower case letters.



Let P and Q be a pair of paths from A to N with no edge in common, and suppose that P contains the edge a and Q contains the edge b.

(a) By considering vertex degrees, explain why the paths P and Q must have no vertex in common, other than A and N. [2]

(b) Explain why P must contain the edge s and Q must contain the edge t. [3]

(c) If one of the paths P and Q contains the vertex F and the other contains the vertex I, explain why P must contain F and Q must contain I. [3]

(d) Write down all possible paths of length 7 from A to N that pass through both the vertices B and F, and all possible paths of length 7 from A to N that pass through both the vertices C and I. [3]

(e) Explain why there are precisely two Hamiltonian cycles in the graph and find them both. [4]

Question 13

The distances (in miles) between six towns A, B, C, D, E and F are as follows:

	A	B	C	D	E	F
A	–	32	42	34	17	20
B	32	–	18	15	32	40
C	42	18	–	12	27	32
D	34	15	12	–	22	30
E	17	32	27	22	–	13
F	20	40	32	30	13	–

(a) Use Kruskal's algorithm to find a minimum connector connecting these towns. What is its total length?
(You must demonstrate that you are using the algorithm correctly, by clearly listing the edges in the order in which you choose them.) [6]

(b) Consider the travelling salesman problem for these six towns. Use Kruskal's algorithm to find *three* lower bounds for the solution to this travelling salesman problem. [8]

(c) Which is the best lower bound that you found in part (b)?
(No explanation required.) [1]

Question 14 (essay question)

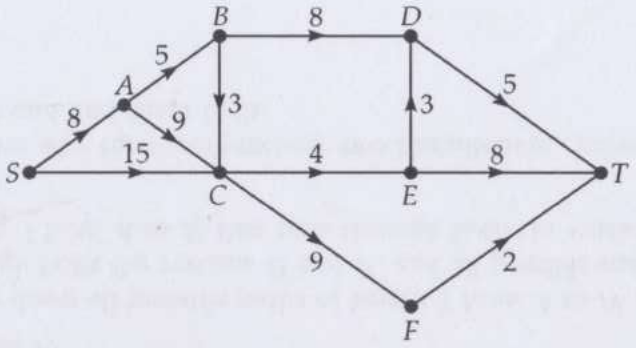
Write a short essay on *duality in planar graphs.*

[Your essay should be written in clear continuous prose. You may include diagrams and equations, but they should be clearly explained. Your essay should be in the order of 300-500 words.] [15]

SECTION B: NETWORKS

Question 15

Consider the following weighted digraph:



(a) Use the shortest path algorithm to find *all* the shortest paths from S to T. (You must demonstrate that you are using the algorithm correctly, by writing down all the vertex potentials, stating clearly the order in which you find them.) [8]

(b) Suppose that you have a computer program for finding the length s of the shortest path(s) and the length l of the longest path(s) from S to T in a given weighted digraph. There is one edge E whose exact length e is unknown. You run the program twice, giving values of 5 and 10 for e. The results are as follows:

if $e = 5$, then $s = 28$ and $l = 32$;

if $e = 10$, then $s = 31$ and $l = 36$.

(i) Find the largest value of e such that the edge E is on a shortest path.

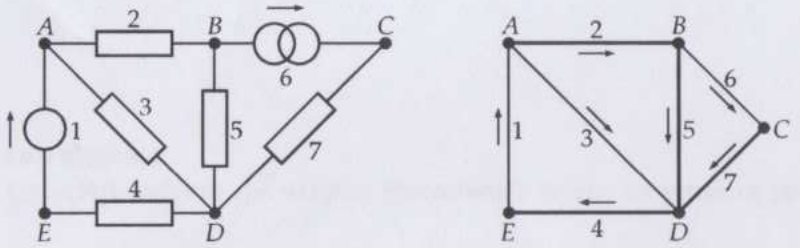
(ii) Find the smallest value of e such that the edge E is on a longest path.

(iii) Find the values of s and l when $e = 7$.

(iv) Write down the values of s and l when $e = 12$. [7]

Question 16

The following diagrams represent an electrical network and a corresponding oriented graph:



(a) The component equations for the two source components (1 and 6) are given as $v_1 = -8$ and $i_6 = 2v_5$. Write down the component equations for the other five components. [2]

(b) Write down the fundamental cutset equations in matrix form. [4]

(c) Given that the fundamental cycle equations associated with the given spanning tree are

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ v_4 \\ v_5 \\ v_7 \\ v_1 \\ v_3 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

write down the **H**-matrix equation. [5]

(d) Describe briefly how your answers to parts (a) and (c) change if component 4 is replaced by an inductor and component 3 is replaced by a capacitor. [4]

Question 17 (essay question)

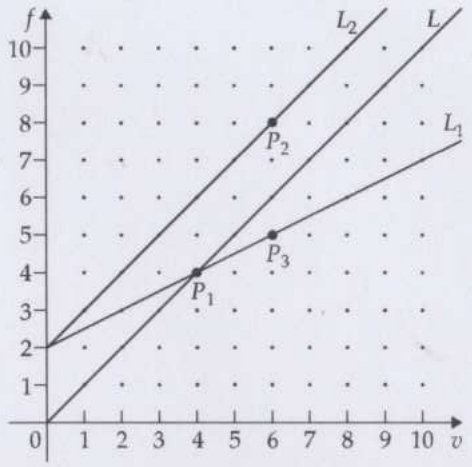
Write a short essay on *some variations of the problem of finding maximum flows in basic networks.*

[Your essay should be written in clear continuous prose. You may include diagrams and equations, but they should be clearly explained. Your essay should be in the order of 300-500 words.] [15]

SECTION C: DESIGN

Question 18

It is possible to represent a polyhedron with v vertices and f faces by a point with integer coordinates (v, f) on a two-dimensional grid of points, as shown below.



(a) On the figure, there are three grid points P_1, P_2 and P_3 marked. For each of these points, write down the number of vertices, faces and edges, and name the corresponding polyhedron. [6]

(b) There are also two lines L_1 and L_2 marked, and two sets of grid points lie on these lines. For each line:

(i) Write down the equation relating the number of vertices and the number of faces;

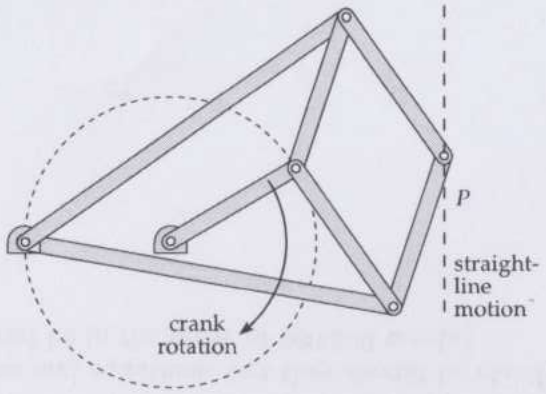
(ii) Name a set of polyhedra all of whose members correspond to grid points on this line;

(iii) Explain which grid points on this line do not correspond to members of this set of polyhedra. [7]

(c) What is the significance of reflecting a grid point in the central line L to give the 'mirror image' grid point? [2]

Question 19

(a) The diagram below shows a Peaucellier cell, the earliest linkage to produce straight-line motion from an input rotary motion. The cell is a planar kinematic system, and all joints are revolutes. As the crank is rotated, the point P moves in a straight line.



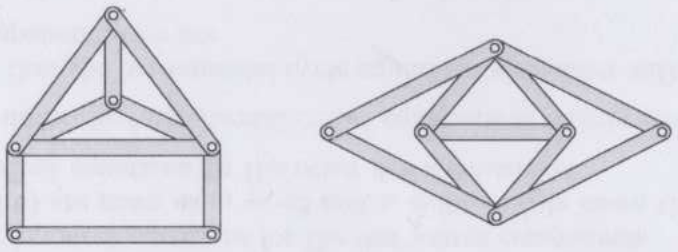
(i) State the number of links and joints of each type in the system.

(ii) If all r-ary joints are expanded and replaced by an appropriate number of binary joints, how many binary joints will there be in the system?
(Justify your answer briefly.)

(iii) Determine the total number of new kinematic systems that can be obtained from the system by expanding each r-ary joint in all valid ways so as to produce systems in which all the joints are binary joints.
(Note that distinct kinematic systems may have isomorphic interchange graphs.)

(iv) Write down the appropriate mobility formula, and hence derive the mobility of the system. [9]

(b) Calculate the mobility of each of the following planar kinematic systems:



(i) (ii)

Carefully explain the evident discrepancy in the mobility of these two systems. [6]

Question 20 (essay question)

Write a short essay on *some uses of latin squares in the design of experiments.*

[Your essay should be written in clear continuous prose. You may include diagrams and equations, but they should be clearly explained. Your essay should be in the order of 300-500 words.] [*15*]

[END OF QUESTION PAPER]

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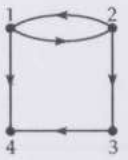
MT365 Solutions to the Specimen Examination Paper

Part 1 (55 marks: 5 marks per question)

Question 1

(a)



2

(b) The digraph has out-degree sequence $(0, 1, 2, 2)$ $\frac{1}{2}$

and in-degree sequence $(1, 1, 1, 2)$. $\frac{1}{2}$

(c) Sum of out-degrees $= 0 + 1 + 2 + 2 = 5$; $\frac{1}{2}$

sum of in-degrees $= 1 + 1 + 1 + 2 = 5$; $\frac{1}{2}$

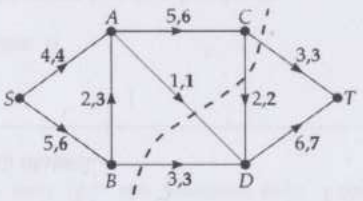
number of arcs $= 5$. $\frac{1}{2}$

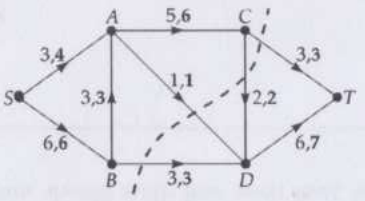
These numbers are all the same, so the handshaking dilemma holds in this case. $\frac{1}{2}$

Remark The answers to parts (b) and (c) can be deduced directly from the incidence matrix given in part (a). [5]

Question 2

(a) There are two maximum flows of value 9:





2

(b) A minimum cut of capacity 9 is $\{BD, AD, CD, CT\}$. 3

Remark You should draw only ONE of the above diagrams. [5]

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Question 3

(a) $\langle Q, R, U \rangle$ is the triangle QRU. $\frac{1}{2} + \frac{1}{2}$

(b) $\langle U, T \rangle$ is the line segment UT. $\frac{1}{2} + \frac{1}{2}$

(c) $\langle P, Q, R, S, T \rangle$ is the tetrahedron $PQRS$. $\frac{1}{2} + \frac{1}{2}$

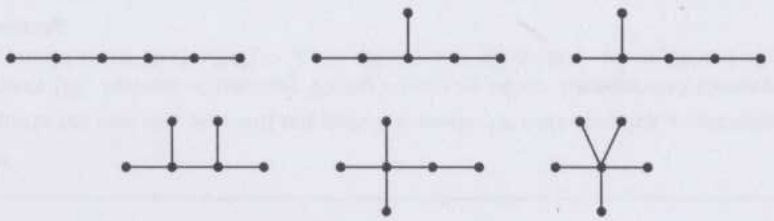
(d) $\langle R, S, T, Q \rangle$ is the tetrahedron $QRST$. $\frac{1}{2} + \frac{1}{2}$

(e) $\langle Q, R, S, U \rangle$ is the triangle QRS. $\frac{1}{2} + \frac{1}{2}$

Remark To gain full marks you must state *which* triangle/line/tetrahedron is relevant. [5]

Question 4

(a) The six non-isomorphic unlabelled trees with 6 vertices are:



3

(b) The second, third and sixth trees above are all central. 1

(c) By Cayley's theorem, the number of labelled trees with 6 vertices is 6^4 $(= 1296)$. $\frac{1}{2} + \frac{1}{2}$

Remarks

- In part (a), there is a penalty of $\frac{1}{2}$ mark for each tree omitted, or included more than once.
- In part (b) it is necessary to indicate just one tree that is central.
- In part (c), it is not necessary to calculate the numerical value of 6^4. There is $\frac{1}{2}$ a mark for mentioning Cayley's theorem, and $\frac{1}{2}$ a mark for the expression 6^4 (or for 1296). [5]

Question 5

(a) (1) 12 weeks 1

(2) START $\rightarrow C \rightarrow E \rightarrow H \rightarrow$ FINISH 1

(b) The float of each activity is

(latest starting time) − (earliest starting time). 1

Thus the float of activity D is $5 - 2 = 3$ weeks, 1

and the float of activity B is $2 - 0 = 2$ weeks. 1

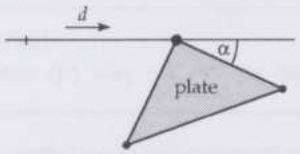
Remarks

- In part (a), the question says 'Write down', so no explanation is required.
- In part (b), the question says 'Find', so you should state how each float is calculated. [5]

Question 6

(a) The plate has *two* freedoms. 1

(b) There are various possibilities — for example, the distance d along the given line of the constrained corner from a fixed point on the given line, and the angle α between the line and a side of the triangle. 2



(c) *Two* further constraints are needed to immobilize the plate. 1

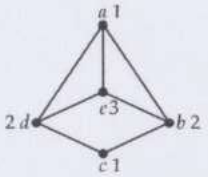
Possible ways of imposing these constraints include fixing both coordinates of some other point, or imposing one constraint on each of two points (one of which may be the corner that lies on the given line). 1

Remark In answering a question of this kind, you may save time by drawing a diagram. [5]

Question 7

(a) Since G contains a triangle, $\chi(G) \geq 3$.

An explicit 3-colouring of the vertices of G is



Thus $\chi(G) \leq 3$.

It follows from the above two inequalities that $\chi(G) = 3$. 1

(b) The maximum vertex degree of G is 3.

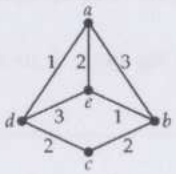
It follows from Vizing's theorem that $\chi'(G)$ is 3 or 4. 1

We show that $\chi'(G) \neq 3$ by trying to construct a 3-edge colouring.

The edges ad, ae and ab must be coloured differently, so we colour them 1, 2, 3, respectively.

Then the edge de must be coloured 3, and the edge be must be coloured 1.

Then both the edges bc and cd must be coloured 2, which is impossible. $1\frac{1}{2}$



It follows that $\chi'(G) = 4$. $\frac{1}{2}$

(c) Ind G is 2. 1

Remarks

- In parts (a) and (b), you will not gain full marks by only showing a colouring.
- In part (b), instead of quoting Vizing's (or any other appropriate) theorem, you could show that $\chi'(G) > 3$, as above, and then show an explicit 4-edge colouring. [5]

Question 8

(a) A maximum matching with 4 edges exists; for example, 1

$x_1y_1, x_2y_2, x_3y_3, x_4y_5$. 1

(b) There is no maximum matching with 4 edges, because the subset $\{x_1, x_3, x_4\}$ of X does not satisfy the marriage condition — the 3 vertices x_1, x_3 and x_4 in X are joined to only 2 vertices y_3 and y_5 in Y. 1+1 1

Remark In part (a), there are three other matchings with 4 edges, each including the edge x_4y_5. [5]

Question 9

(a) The code is linear, $\frac{1}{2}$

because the sum of any two codewords of C is a codeword of C. $\frac{1}{2}$

(b) The code is not cyclic; $\frac{1}{2}$

for example, 1100 is not a codeword of C. $\frac{1}{2}$

(c) A parity check matrix for C is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \quad 1$$

The codewords of the dual code C^* are the sums of rows of $\mathbf{H}$, so

$$C^* = \{0000, 1001, 0110, 1111\} = C. \quad 1$$

Thus C is self-dual. 1

Remarks

- In part (a), it is not necessary to write down the sum of each pair of codewords.
- In part (b), you should write down a counter-example.
- In part (c), an alternative explanation is: 'The matrix $\mathbf{H}$ is both a parity check matrix and a generator matrix for C, so C is self-dual'. [5]

Question 10

(a) When $n = 4$, we have

$$T_1(n) = 4\log_2 4 + 2(4^{3/2}) = 8 + 16 = 24 \quad 1$$

and

$$T_2(n) = 4^2 = 16, \quad 1$$

so T_2 is the faster algorithm when $n = 4$. 1

(b) When n is large, the function n^2 dominates both the functions $n\log_2 n$ and $2n^{3/2}$, so T_1 is the faster algorithm when n is large. 1 1

Remark The question says 'determine', so some justification must be given. [5]

Question 12

(a) The table of differences is

	0	2	3	7
0	0	11	10	6
2	2	0	12	8
3	3	1	0	9
7	7	5	4	0

1

Each non-zero number appears exactly once in the table, so $\{0, 2, 3, 7\}$ is a perfect difference set. 1

(b) The first four blocks of the cyclic design are

1	2	3	4
0	1	2	3
2	3	4	5
3	4	5	6
7	8	9	10

1

(c) $\lambda = 1$, $v = 13$, $k = 4$ and $r = 4$, so 1

$$\lambda(v-1) = 1 \times 12 = 12 \quad \text{and} \quad r(k-1) = 4 \times 3 = 12. \qquad 1$$

Hence $\lambda(v-1) = r(k-1)$.

[5]

Part 2 (45 marks: 15 marks per question)

SECTION A: GRAPHS

Question 12

(a) If the paths P and Q share a vertex (other than A or N), this vertex must have degree 4 or more, since P and Q have no common edge. 1

Since there is no such vertex, P and Q have no vertices in common, other than A and N. 1

(b) If P does not contain the edge s, then P must contain the edges a and t, and so Q must contain the edges b and s. Thus the paths P and Q must cross, contradicting the fact that they have no common vertex other than A and N [the result of part (a)]. So P must contain s. 2

A similar argument shows that Q must contain t. 1

(c) If P does not contain the vertex F, then P must contain the vertices B and I, and Q must contain the vertices C and F. Thus the paths P and Q must cross, contradicting the result of part (a). So P must contain F, and Q must contain I. 3

(d) 1. $ABDEGFLN$ $(acfhjns)$ $\frac{1}{2}$
2. $ABFGJKLN$ $(acjloqs)$ $\frac{1}{2}$
3. $ABFGJKMN$ $(aejlort)$ $\frac{1}{2}$
4. $ACDEHIMN$ $(bdfikpt)$ $\frac{1}{2}$
5. $ACIHJKLN$ $(bgkmoqs)$ $\frac{1}{2}$
6. $ACIHJKMN$ $(bgkmort)$ $\frac{1}{2}$

(e) From parts (a) to (d) a Hamiltonian cycle will comprise path P, from A to N, then path Q in reverse, from N to A, such that P passes through B, F and L, and Q through M, I and C. 1

For these paths to pass through G and H, respectively, they must both be of length 7 and there are only two such options: 1

$ABDEGFLNMKJHICA$ $(acfhjnstromkgb)$ (from paths 1 and 6) 1

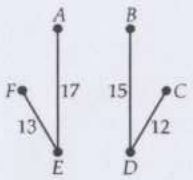
$ABFGJKLNMIHEDCA$ $(aejloqstpkifdb)$ (from paths 2 and 4) 1

[15]

Question 13

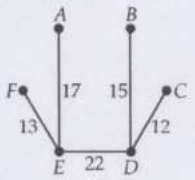
(a) Choose edges in the following order:

CD	(length 12)	1
EF	(length 13)	1
BD	(length 15)	1
AE	(length 17)	1



We cannot now choose BC (length 18) or AF (length 20), as each of these edges would create a cycle.

We therefore choose DE (length 22). 1



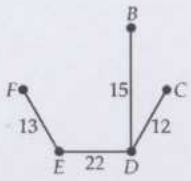
This gives the following minimum connector with total length

$12 + 13 + 15 + 17 + 22 = 79$ miles. 1

(b) We remove a vertex, find the length of a minimum connector for the remaining vertices, and add the lengths of the two shortest edges incident with the removed vertex. 2

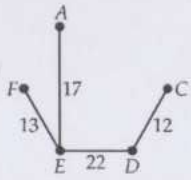
This gives the following lower bounds:

vertex A



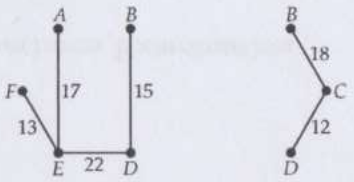
total distance $\geq$ $(12 + 13 + 15 + 22)$ + $(17 + 20) = 99$ miles

vertex B



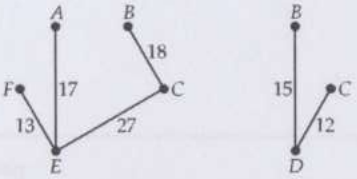
total distance $\geq$ $(12 + 13 + 17 + 22)$ + $(15 + 18) = 97$ miles

vertex C



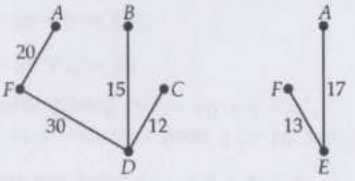
total distance ≥ (13 + 15 + 17 + 22) + (12 + 18) = 97 miles

vertex D



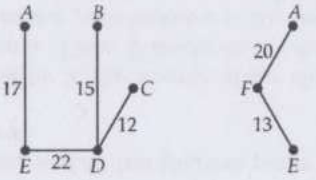
total distance ≥ (13 + 17 + 18 + 27) + (12 + 15) = 102 miles

vertex E



total distance ≥ (12 + 15 + 20 + 30) + (13 + 17) = 107 miles

vertex F



total distance ≥ (12 + 15 + 17 + 22) + (13 + 20) = 99 miles

6 (2 marks for each of three trees)

(c) The best lower bound obtained is the *highest* of these bounds found. 1

Remark In part (b), only three of the above trees are asked for. [15]

Question 14

(essay question)

Your account should include most of the following:

- an explanation of the term *dual graph*;
- an example of a dual graph;
- if G^* is a dual of a connected planar graph G, then G is a dual of G^* — that is, $(G^*)^*$ is isomorphic to G;
- the number of vertices, edges and faces of G^*;
- the relevance of duality to colouring the faces of maps and the vertices of planar graphs;
- duality properties of cycles and cutsets in graphs. 15

[15]

SECTION B: NETWORKS

Question 15

(a) We start by giving S zero potential. Using the tabular method, we obtain the following potentials (shown in boxes):

iteration	origin vertex	vertices assigned labels						
		A	B	C	D	E	F	T
1	S	[8]		15				
2	A		[13]					
3	B			[15]	21			
4	C					[19]	24	
5	E				[21]			27
6	D						[24]	26
7	F							[26]

6

Tracing back through the table, we find *two* shortest paths of length 26:

$SABDT$ and $SCFT$. 2

(b) As the length of E is increased by 5, the lengths of the shortest and longest paths increase, but by less than 5. Thus, E starts on the shortest path (when $e = 5$) and drops out of the shortest path before $e = 10$; also, E is *not* on the longest path when $e = 5$, but comes onto the longest path at some point before $e = 10$. 2

Thus we deduce the following.

(i) Since s increases by 3 as e increases from 5 to 10, the largest value of e such that E is on a shortest path is $e = 5 + 3 = 8$. 1

(ii) Since l increases by 4 as e increases from 5 to 10, the smallest value of e such that E is on a longest path is $e = 10 - 4 = 6$. 1

(iii) When $e = 7$, then $s = 28 + 2 = 30$. 1

When $e = 7$, then $l = 36 - 3 = 33$. 1

(iii) When $e = 12$, $s = 31$ and $l = 38$. 1

Remark In part (a), you do not need to use the tabular method. If you do, you need not give the complete table, but you should include enough of it to show that you are using the correct method. [15]

Question 16

(a) The five components are all resistors satisfying Ohm's law:

$$v_2 = R_2 i_2, \quad v_3 = R_3 i_3, \quad v_4 = R_4 i_4, \quad v_5 = R_5 i_5, \quad v_7 = R_7 i_7. \qquad 2$$

(b) The fundamental cutset equations are written in matrix form as follows:

(Errata)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_4 \\ i_5 \\ i_7 \\ i_1 \\ i_3 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \not{0} \\ \not{0} \end{bmatrix} \qquad 4$$

(c) The **H**-matrix equation $\mathbf{Hx} = \mathbf{y}$ is

$$\left[\begin{array}{ccccccc:ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hdashline 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} v_2 \\ v_4 \\ v_5 \\ v_7 \\ v_1 \\ v_3 \\ v_6 \\ \hdashline i_2 \\ i_4 \\ i_5 \\ \hdashline i_7 \\ i_1 \\ i_3 \\ i_6 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -8 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$\mathbf{H} \qquad \mathbf{x} = \mathbf{y}. \qquad 5$$

(Note that line 7, representing the equation $2v_5 - i_6 = 0$, could equally well have -2 and 1, or 1 and $-\frac{1}{2}$. instead of 2 and -1.)

(d) The component equations for components 3 and 4 would be replaced by

$$i_3 = C\frac{dv_3}{dt} \quad \text{and} \quad v_4 = L\frac{di_4}{dt}. \qquad 2$$

The matrix equation would be replaced by the equation

$$\mathbf{Hx} = \mathbf{y} + \mathbf{K}\dot{\mathbf{x}},$$

where the vector $\mathbf{x}$ has v_3 and i_4 appearing last, the rows of $\mathbf{H}$ are correspondingly rearranged, $\mathbf{K}$ is another 14×14 matrix, and $\dot{\mathbf{x}}$ is a 14×1 vector involving the derivatives $\dot{v}_3$ and $\dot{i}_4$. 2

[15]

Question 17

(essay question)

Your account should include most of the following:

- a description of the network flow problem for basic networks;
- the adjustments for non-basic networks with several sources and sinks, undirected arcs, and/or capacity restrictions on the vertices;
- the max-flow min-cut theorem for basic networks, and its connections with Menger's theorem;
- a brief discussion of networks with lower and upper capacities. 15

[15]

SECTION C: DESIGN

Question 18

(a) For each point, the number of edges can be calculated by Euler's formula $v - e + f = 2$:

P_1 $v = 4$, $f = 4$, $e = 6$ (tetrahedron) 2

P_2 $v = 6$, $f = 8$, $e = 12$ (octahedron) 2

P_3 $v = 6$, $f = 5$, $e = 9$ (triangular prism). 2

(b) L_1:

(i) The relationship between v and f is $v = 2f - 4$. 1

(ii) The *prisms* all correspond to grid points on this line. 1

(Errata) (iii) The grid points on this line with $v < 6$ don't correspond to a prism; that is,$(0, 2)$, $(2, 3)$ and $(4, 4)$. 1

L_2:

(i) The relationship between v and f is $f = v + 2$. 1

(ii) The *anti-prisms* all correspond to grid points on this line. 1

(iii) The grid points on this line with $v < 6$ 1

and those with v odd don't correspond to an anti-prism. 1

(c) Reflecting a grid point in the line L results in the numbers v and f being interchanged. 1

This corresponds to the process of transforming a polyhedron into its dual. 1

Remark Do not be put off by the fact that the question looks unfamiliar; it is in fact, quite straightforward. [15]

Question 19

(a) (i) There are 8 binary links (including the base), 2 binary joints and 4 ternary joints. 2

(ii) Each ternary joint expands to 2 binary joints, so the 4 ternary joints become 8 binary joints. So there will be 10 binary joints in the system. 2

(iii) Each r-ary joint can be expanded to give an equivalent system of $r-1$ binary joints, and this can be done in r^{r-2} different ways. Thus, each of the 2 binary joints is unchanged, and each of the 4 ternary joints can be expanded in $3^{3-2}=3$ ways. The total number of expansions into binary joints is, therefore, $1\times 1\times 3\times 3\times 3\times 3=81$. 3

(iv) For a planar system with 8 links and 10 binary joints, we have

$$M=3(n-1)-2j=3\times(8-1)-(2\times 10)=1.$$ 2

Alternatively, in terms of 6 multiple joints and 8 binary links, we have

$$M=2g-n_2-3=(2\times 6)-8-3=1.$$

(b) Each system has 9 binary links, 2 binary joints, 2 ternary joints, and 2 quaternary joints.

Each ternary joint expands to 2 binary joints and each quaternary joint expands to 3 binary joints, so there are $2+(2\times 2)+(2\times 3)=12$ binary joints in the system. Thus, for each system,

$$M=3(n-1)-2j=3\times(9-1)-(2\times 12)=0.$$ 3

The discrepancy in the mobility arises from special geometry because only one system is actually rigid. 1

System (i) is not rigid: the top part is overconstrained, while the bottom part is free to move.

System (ii) is rigid. 2

Remark Explain clearly what you are doing at each stage; then if you get the numbers wrong [for example, if you forget the base in part(a)], you will still gain marks for using the correct method. [15]

Question 20

(essay question)

Your account should include most of the following:

- the definition of a latin square;
- some general descriptions of the use of latin squares (as outlined on pages 39–40 of *Design 4*);
- the use of orthogonal latin squares in the construction of replicates for the square lattice design. 15

[15]